

Endoscopic Transfer between Eigenvarieties for definite Unitary groups

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Abstract

In this paper, we extend the endoscopic transfer of definite unitary group $U(n)$, which sends a pair of automorphic forms of $U(n_1), U(n_2)$ to an automorphic form of $U(n_1 + n_2)$, to finite slope p -adic automorphic forms for definite unitary groups by constructing a rigid analytic map between eigenvarieties $\mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_{(n_1+n_2)}$, which at classical points interpolates endoscopic transfer map.

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1 Introduction

Let us fix a rational prime p and embeddings,

$$\iota_p : \bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p, \text{ and } \iota_\infty : \bar{\mathbb{Q}} \rightarrow \mathbb{C}.$$

For any prime ℓ , we also fix an embedding of $W_{\mathbb{Q}_\ell} \hookrightarrow W_{\mathbb{Q}}$, where W_F denotes the Weil group of F . Let E/\mathbb{Q} be a quadratic imaginary field and G/\mathbb{Q} be a unitary group in $n \geq 1$ variables attached to E/\mathbb{Q} . We assume that $G(\mathbb{R})$ is the compact real unitary group (that is G is definite), and that $G(\mathbb{Q}_p) \simeq GL_n(\mathbb{Q}_p)$. We fix such a G and denote it by definite unitary group $U(n)$.

By recent work of Clozel, Harris, Labesse, Ngô and many other mathematicians, we made significant progress in the Langlands program for the unitary group $U(n)$. In particular we know how to attach a Galois representation ρ_π to an automorphic form π of $U(n)$. Since the fundamental lemma for the unitary group is known, in principle we know in most cases the classical endoscopic transfer for $U(n)$

(for $n \leq 3$, this is due to Rogawski([11], in general see [7], and volume 2). Let n_1, n_2 be two natural numbers and $n = n_1 + n_2$, then $U(n_1) \times U(n_2)$ is an endoscopic subgroup of $U(n)$. Let π_1 and π_2 be automorphic representations of $U(n_1)$ and $U(n_2)$ respectively. Then the endoscopic transfer of (π_1, π_2) is an automorphic representation π of $U(n)$. If $\rho_{\pi_1}, \rho_{\pi_2}, \rho_{\pi}$ are the Galois representations associated to π_1, π_2, π respectively, then we have $\rho_{\pi} = \rho_{\pi_1} \oplus \rho_{\pi_2}$.

Another aspect of study of automorphic forms, initiated by Serre, is that, a prime number p being chosen, the notion of classical automorphic forms can be extended to the notion of p -adic automorphic forms. Using ordinary p -adic automorphic forms, Hida constructed family of ordinary p -adic automorphic forms, known as Hida family, for various reductive groups. Coleman and Mazur [8], extended Hida's work to non-ordinary p -adic modular forms, and constructed tame level 1 eigencurve. Buzzard [4] formalized the construction of Coleman and Mazur, and gave a machinery, called the "Eigenvariety Machine", which gives a p -adic family of automorphic forms, known as the eigenvariety, as output. He used it to construct tame level N eigencurve and eigenvariety for Hilbert modular forms. Using Buzzard's eigenvariety machine, Chenevier [5] and Emerton [9], separately constructed eigenvariety for definite unitary group $U(n)$, which we denote by \mathcal{E}_n . The eigenvariety \mathcal{E}_n is a reduced rigid analytic space, which is equidimensional of dimension n .

In this paper, we construct p -adic endoscopic transfer map for definite unitary group $U(n)$. Similar work was done by Chenevier [6], where he constructed the p -adic Jacques-Langlands correspondence for GL_2 . We follow a similar approach. More precisely, let \mathcal{E}_n denote the eigenvariety associated to definite unitary group $U(n)$, as constructed by Chenevier. In the eigenvariety \mathcal{E}_n , every automorphic form π of $U(n)$ appears not once but (roughly) $n!$ times as a point of \mathcal{E}_n , each time augmented with a little combinatorial structure called a refinement \mathcal{R} , which is an ordering of the eigenvalues of semi-simple conjugacy class associated to π_p . We construct a rigid analytic map

$$f : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n$$

which at "nice" classical points interpolate classical endoscopic transfer. Further we show that such a map is unique.

Theorem A. There exists a unique map of eigenvarieties

$$f_{(n_1, n_2)} : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n$$

where $n = n_1 + n_2$, such that a "nice" classical point of $\mathcal{E}_{n_1} \times \mathcal{E}_{n_2}$, $((\pi_1, \mathcal{R}_1), (\pi_2, \mathcal{R}_2))$ maps to a classical point of \mathcal{E}_n , (π, \mathcal{R}) , where π is the endoscopic transfer of π_1 and π_2 and $\mathcal{R} = (\mu_1(Frob_p)\mathcal{R}_1, \mu_2(Frob_p)\mathcal{R}_2)$, here μ_1 and μ_2 as in (5).

We call the image of the map $f_{(n_1, n_2)}$ the endoscopic component of type (n_1, n_2) in \mathcal{E}_n , and denote it by $\mathcal{E}_{(n_1, n_2)}$. In particular, we define p -adic endoscopic transfer π for any pair of p -adic automorphic forms (π_1, π_2) of the unitary groups $U(n_1), U(n_2)$ as the image of (π_1, π_2) under the map $f_{(n_1, n_2)}$.

Key ingredient of our proof is a comparison theorem due to Chenevier (see Theorem 4.4), which allows us to construct the map, by constructing maps between the fibers of the eigenvarieties over Zariski dense set of points in the weight space \mathcal{W} .

Plan for the paper: In section 2, of the paper we briefly recall the construction of the eigenvariety \mathcal{E}_n , and a few of it's relevant properties. In section 3, we review some known (and expected) results about the classical endoscopic transfer of automorphic forms of definite unitary group $U(n)$. In the final section, we give the construction of the rigid analytic map between the eigenvarieties $\mathcal{E}_{n_1} \times \mathcal{E}_{n_2}$ and \mathcal{E}_n .

2 Brief review of the construction of eigenvariety for $U(n)$

In this section we briefly recall the construction of Eigenvariety for $U(n)$ due to Chenevier. For details we refer readers to [2, Chapter 7], [5].

2.1 Construction

The eigenvariety depends on the choice of Hecke algebra H , so first we fix a Hecke algebra for our purpose. We fix a subset S_0 (which have Dirichlet density 1) of primes l ($l \neq p$) split in E , such that $G(\mathbb{Q}_l) \simeq GL_n(\mathbb{Q}_l)$ and $G(\mathbb{Z}_l)$ is a maximal compact subgroup. We set $\mathcal{H}^{ur} = \otimes_{l \in S_0} \mathcal{H}_l$, where $\mathcal{H}_l \simeq \mathbb{Q}_p[T_{1,l}, \dots, T_{n,l}, T_{1,l}^{-1}, \dots, T_{n,l}^{-1}]^{S_n}$ is the Hecke algebra for $GL_n(\mathbb{Q}_l)$, with respect to its maximal compact subgroup $G(\mathbb{Z}_l)$. Let T be the diagonal torus of $G(\mathbb{Q}_p) \cong GL_n(\mathbb{Q}_p)$, and $T^0 = T \cap GL_n(\mathbb{Z}_p)$. Let $U \simeq T/T^0$ be the subgroup of diagonal matrices whose entries are integral powers of p , $U^- \subset U$ be the submonoid whose elements have the form $\text{diag}(p^{a_1}, \dots, p^{a_n})$, with $a_i \in \mathbb{Z}$ and $a_i \geq a_{i+1}$ for all i . Let \mathcal{A}_p^- be the subring of Hecke-Iwahori algebra generated by $u \in U^-$ and \mathcal{A}_p be the Atkin-Lehner algebra contained in the Hecke-Iwahori algebra of $G(\mathbb{Q}_p) \simeq GL_n(\mathbb{Q}_p)$, with coefficient in \mathbb{Q}_p . Then $\mathcal{A}_p \cong \mathbb{Q}_p[U]$ and $\mathcal{A}_p^- \cong \mathbb{Q}_p[U^-]$ (see [2, Prop. 6.4.1]). Let us define $\mathcal{H} := \mathcal{A}_p \otimes \mathcal{H}^{ur}$ to be the Hecke Algebra. Let us also define $\mathcal{H}^- := \mathcal{A}_p^- \otimes \mathcal{H}^{ur}$.

The weight space \mathcal{W}_n is the rigid analytic space over \mathbb{Q}_p defined as $\mathcal{W}_n := \text{Hom}_{\text{group}, \text{cts}}(T^0, \mathbb{G}_m^{rig})$. It is isomorphic to finite disjoint union of unit open n -dimensional balls. We see \mathbb{Z}^n embedded in \mathcal{W}_n via the map $(k_1, \dots, k_n) \mapsto ((x_1, \dots, x_n) \mapsto x_1^{k_1} \dots x_n^{k_n})$. Let $V \subset \mathcal{W}_n$ be an affinoid open. Then Chenevier defined the space of p -adic automorphic forms of weight in V , radius of convergence r , $\mathcal{S}(V, r)$ ([2, Section 7.3.4]) as follows: Let I denote the Iwahori subgroup of $GL_n(\mathbb{Q}_p)$, and B the standard Borel subgroup. Let \tilde{N}_0 be the subgroup of lower triangular matrices of I . Let $\chi : T^0 \rightarrow \mathcal{O}(\mathcal{W})^*$ denote the tautological character and $\chi_V : T^0 \rightarrow A(V)^*$ the induced continuous character. For $r \geq r_V$, define the space of functions:

$$\mathcal{C}(V, r) = \{f : IB \rightarrow A(V) \mid f(xb) = \chi_V(b)f(x), \forall x \in IB, b \in B, f|_{\tilde{N}_0} \text{ is } r \text{ analytic}\}.$$

$\mathcal{C}(V, r)$ is a $M := IU^-I$ module. For a M -module E , define an \mathcal{H}^- -module $F(E)$ as:

$$F(E) = \{f : G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \rightarrow E \mid f(g(1 \times k_p)) = k_p^{-1} f(g), \forall g \in G(\mathbb{A}_f), k_p \in I, f \text{ is smooth outside } p\}.$$

We define the space of p -adic automorphic forms of weights in V , radius of convergence r as the \mathcal{H}^- -module $\mathcal{S}(V, r) := F(\mathcal{C}(V, r))$. $\mathcal{S}(V, r)$ is an ortho-normalizable Banach $A(V)$ -module with property (Pr) (as in [4]). It is equipped with an $A(V)$ linear action of $\mathcal{H}^- = \mathcal{A}_p^- \otimes \mathcal{H}^{ur}$. The modules $\{\mathcal{S}(V, r), V, r \geq r_V\}$ are compatible in a sense defined there.

If \underline{k} is a point in the weight space, the above construction remains valid, we get a module $M_{(\underline{k})} = \mathcal{S}(\underline{k}, r)$. Moreover if $\underline{k} \in \mathbb{Z}^{n,-}$, the choice of an highest-weight vector in $W_{\underline{k}}(\mathbb{Q}_p)$ with respect to the standard Borel B gives a natural \mathcal{H}^- -equivariant inclusion:

$$F(W_{\underline{k}}(L)^*) \otimes \delta_{\underline{k}} \hookrightarrow \mathcal{S}(\underline{k}, 0),$$

where $\delta_{\underline{k}}$ is a character of U obtained by restricting $W_{\underline{k}}(\mathbb{Q}_p)$ to U . Image of the map is referred as the space of classical p -adic automorphic forms of weight \underline{k} .

Let $u_0 = \text{diag}(p^{n-1}, p^{n-2}, \dots, p, 1)$. Then u_0 acts compactly on the (\mathbf{Pr}) -family of Banach modules $\{\mathcal{S}(V, r), V, r \geq r_V\}$. Let $P = P_{u_0}$ be the Fredholm series associated to u_0 . Then for $r \geq r_V$, we have,

$$P(T)|_V = \det(1 - Tu_0|_{\mathcal{S}(V, r)}) \in 1 + TA(V)\{\{T\}\}.$$

Fix a $\nu \in \mathbb{R}$ such that $V(P(T))$ is adapted to ν (see [1, Definition II.1.8], also see [4]). Then in $A(V)\{\{T\}\}$, P has a unique factorization $P = QR$, with $Q \in 1 + TA(V)[T]$ and we have a decomposition of $A(V)$ Banach module $\mathcal{S}(V, r)$ as,

$$\mathcal{S}(V, r) = M(V)^{\leq \nu} \oplus N(V, r) \quad (1)$$

which is \mathcal{H}^- stable and such that:

- $M(V)^{\leq \nu}$ is a finite projective $A(V)$ -module which is independent of $r \geq r_V$.
- The characteristic polynomial of u_0 on $M(V)^{\leq \nu}$ is $Q^{rec}(T)$, the reciprocal polynomial on Q , and $Q^{rec}(u_0)$ is invertible on $N(V, r)$.

In particular the eigenvalues of U_p on $M(V)^{\leq \nu}$ has valuation less than or equal to ν . When V is a point \underline{k} , we similarly get a finite projective module $M_{\underline{k}}^{\leq \nu}$.

The local piece of the eigenvariety is by definition the maximal spectrum of the $A(V)$ -algebra generated by the image of $\mathcal{H} = \mathcal{H}^-[u_0]^{-1}$ in $\text{End}_{A(V)}(M(V)^{\leq \nu})$. By Buzzard's eigenvariety machine (see [4]) these pieces glue together to form the eigenvariety (for details see [2, Section 7.3.6]), which we will denote by \mathcal{E}_n . The eigenvariety is reduced p -adic analytic space \mathcal{E}_n , equipped with a ring homomorphism $\psi : \mathcal{H} \rightarrow \mathcal{O}(\mathcal{E}_n)^{rig}$, and an analytic map $\omega : \mathcal{E}_n \rightarrow \mathcal{W}$.

2.2 Points on the eigenvariety

Let $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $k_1 \geq k_2 \geq \dots \geq k_n$. A p -refined automorphic representation of weight \underline{k} is a tuple (π, \mathcal{R}) such that,

- π is an irreducible automorphic representation of G .
- $\pi_\infty \simeq_{\iota_\infty} W_{\underline{k}}(\mathbb{C})$.
- π_p is unramified and \mathcal{R} is an accessible refinement of π_p . [2, Section 6.4.4]

Let us recall that an refinement of π_p is an ordering

$$\mathcal{R} = (\phi_1, \dots, \phi_n)$$

of the eigenvalues of the Langlands conjugacy associated to π_p . Equivalently, it is a character $\chi : U \rightarrow \mathbb{C}^*$, sending $(1, \dots, 1, p, 1, \dots, 1)$ to ϕ_i . It is called accessible if $\chi \delta_B^{-1/2}$ occurs in π_p^I , here B is the standard Borel subgroup, I is the Iwahori subgroup and δ_B is the modulus character.

A p -refined automorphic representation of weight \underline{k} is a tuple (π, \mathcal{R}) , which is viewed as a point in the eigenvariety in the following manner:

Let $\delta_{\underline{k}} : U \rightarrow p^{\mathbb{Z}}$ be the character sending $\text{diag}(u_1, \dots, u_n) \rightarrow u_1^{k_1} \dots u_n^{k_n}$. Then there is a unique ring homomorphism $\psi_p : \mathcal{A}_p \rightarrow \mathbb{C}$, such that

$$\psi_{p|U} = \chi \delta_B^{-1/2} \delta_{\underline{k}}. \quad (2)$$

Moreover we also have a ring homomorphism, $\psi_{ur} : \mathcal{H}^{ur} \rightarrow \mathbb{C}$, since $\pi^{G(\hat{\mathbb{Z}}_{S_0})}$ is one dimensional. Hence we have a system of \mathcal{H} eigenvalue $\psi_p \otimes \psi_{ur}$. It is in-fact $\bar{\mathbb{Q}}$ -valued. To the p -refined automorphic representation (π, \mathcal{R}) of weight \underline{k} , we associate a $\bar{\mathbb{Q}}_p$ -valued system of Hecke eigenvalues $\psi_{(\pi, \mathcal{R})} = \iota_p \iota_\infty^{-1}(\psi_p \otimes \psi_{ur})$.

Let $\mathcal{Z}_0 \subset \text{Hom}_{\text{ring}}(\mathcal{H}, \bar{\mathbb{Q}}_p) \times \mathbb{Z}^n$ be the set of pairs $(\psi_{(\pi, \mathcal{R})}, \underline{k})$ associated to all p -refined automorphic form (π, \mathcal{R}) of weight \underline{k} . For a fixed $\mathcal{Z} \subset \mathcal{Z}_0$, the associated eigenvariety \mathcal{E}_n comes with an accumulation and Zariski dense subset $Z \subset \mathcal{E}_n(\bar{\mathbb{Q}}_p)$, such that the natural evaluation map $\mathcal{E}_n(\bar{\mathbb{Q}}_p) \rightarrow \text{Hom}_{\text{ring}}(\mathcal{H}, \bar{\mathbb{Q}}_p)$, sending $x \mapsto \psi_x := (h \mapsto \psi(h)(x))$, induces a bijection between Z and \mathcal{Z} sending $z \mapsto (\psi_z, \omega(z))$. (We might think \mathcal{E}_n as “Zariski closure” of the set \mathcal{Z} .)

2.2.1 Control Theorem

Let $\mathbb{Z}^{n,-} = \{(a_1, \dots, a_n) | a_1 > \dots > a_n\}$. Let $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^{n,-}$. For this section let $V = \underline{k}$, be a closed point. Let $U^{--} = \{\text{diag}(p^{a_1}, \dots, p^{a_n}) | a_i \in \mathbb{Z}, a_1 > \dots > a_n\}$. An element $f \in \mathcal{S}(\underline{k}, r)$ is of finite slope if for some (hence for all) $u \in U^{--}$,

- $L[u].f \subset \mathcal{S}(\underline{k}, r)$ is finite dimensional
- $u|_{L[u].f}$ is invertible.

Hence the finite slope elements form an L -subspace $\mathcal{S}(\underline{k}, r)^{fs}$ of $\mathcal{S}(\underline{k}, r)$, and the \mathcal{A}_p^- module structure on $\mathcal{S}(\underline{k}, r)^{fs}$ extends to \mathcal{A}_p module structure [2, Section 7.3.5]. Note that, $M_{\underline{k}}^{\leq \nu}$ as in (1), is contained in $\mathcal{S}(\underline{k}, r)^{fs}$. We have a natural inclusion of \mathcal{H} -modules $F(W_{\underline{k}}(L)^*) \otimes \delta_{\underline{k}} \subset \mathcal{S}(\underline{k}, r)^{fs} \subset \mathcal{S}(\underline{k}, 0)$, for all $r \in \mathbb{N}$.

The following proposition allows us to determine when a finite slope overconvergent eigenform is classical.

Proposition 2.1. [2, Proposition 7.3.5]/[5, Proposition 4.7.4] *Let $f \in \mathcal{S}(\underline{k}, r)^{fs} \otimes_L \bar{\mathbb{Q}}_p$ be an eigenform for all $u \in U \subset \mathcal{A}_p$. Let $u_0 = \text{diag}(p^n - 1, \dots, p, 1)$, write $u_0(f) = \lambda f$ for $\lambda \in \bar{\mathbb{Q}}_p^*$. If we have,*

$$v(\lambda) < 1 + \min_{i=1}^{n-1} (k_i - k_{i+1}), \quad (3)$$

then f is classical. More precisely, under the same condition the full generalized \mathcal{A}_p -eigenspace of f in $\mathcal{S}(\underline{k}, r)^{fs} \otimes_L \bar{\mathbb{Q}}_p$ is included in the subspace $F(W_{\underline{k}}(L)^) \otimes_L \bar{\mathbb{Q}}_p$ of classical forms.*

3 Endoscopic Transfer for automorphic forms on $U(n)$

In this section we recall known (and expected) results about classical endoscopic transfer for automorphic forms on the definite unitary group $U(n)$.

3.1 Admissible map of L-groups of $U(n)$

Let E be a CM extension of \mathbb{Q} in $\bar{\mathbb{Q}}$. Let $\text{Gal}(E/\mathbb{Q}) = \{1, \sigma\}$. Let $\omega_{E/\mathbb{Q}}$ denote the character of order 2 of idele class group $C_{\mathbb{Q}}$ associated to E/\mathbb{Q} by CFT¹.

The L-group of $U(n)$ is ${}^L U(n) = GL_n(\mathbb{C}) \rtimes W_{\mathbb{Q}}$, where the Weil group $W_{\mathbb{Q}}$ acts on $GL_n(\mathbb{C})$ by its quotient $\text{Gal}(E/\mathbb{Q})$ by $\sigma M \sigma^{-1} := \phi_n(M^{-1})^t \phi_n^{-1}$, where $M \in GL_n(\mathbb{C})$ and $(\phi_n)_{i,j} = (-1)^{n+i} \delta_{i, n+1-j}$. The L-group of $U(n_1) \times \dots \times U(n_r)$ is ${}^L(U(n_1) \times \dots \times U(n_r)) = (GL_{n_1}(\mathbb{C}) \times \dots \times GL_{n_r}(\mathbb{C})) \rtimes W_{\mathbb{Q}}$, where the Weil group $W_{\mathbb{Q}}$ acts on $GL_{n_1}(\mathbb{C}) \times \dots \times GL_{n_r}(\mathbb{C})$ by its quotient $\text{Gal}(E/\mathbb{Q})$.

¹Let $N_{E/\mathbb{Q}} : C_E \rightarrow C_{\mathbb{Q}}$ denotes the norm map, so its image is index 2 subgroup of $C_{\mathbb{Q}}$. Then $\omega_{E/\mathbb{Q}}$ is the nontrivial character of $C_{\mathbb{Q}}$ which is trivial on the image $N_{E/\mathbb{Q}}(C_E)$.

Let $n = n_1 + \cdots + n_r$ be an unordered partion. Following Rogawski, we have an admissible map of L-groups [12, Section 1.2] as below:

$$\xi : {}^L(U(n_1) \times \cdots \times U(n_r)) \rightarrow {}^L U(n) \quad (4)$$

which maps the neutral component of ${}^L(U(n_1) \times \cdots \times U(n_r)) = GL_{n_1} \times \cdots \times GL_{n_r}$ to the subgroup of diagonal blocks of size n_1, \dots, n_r . We fix for once and all a character μ of the idele class group C_E whose restriction to $C_{\mathbb{Q}}$ is $\omega_{E/\mathbb{Q}}$. We regard μ as a character of W_E by means of isomorphism $W_E^{ab} \rightarrow C_E$. Set

$$\mu_j = \begin{cases} \mu & \text{if } n_j \equiv n - 1 \pmod{2}, \\ 1 & \text{otherwise.} \end{cases} \quad (5)$$

and for $w \in W_E$, let

$$\xi(w) = \xi(\mu_1(w)I_{n_1}, \dots, \mu_r(w)I_{n_r}) \times w.$$

To extend ξ to $W_{\mathbb{Q}}$, it suffices to define $\xi(w_{\sigma})$, where $w_{\sigma} \in W_{\mathbb{Q}}$ is a fixed element whose projection to $Gal(E/F)$ is σ . We define

$$\xi(w_{\sigma}) = \xi(\Phi_{n_1}, \dots, \Phi_{n_r})\Phi_n^{-1} \times w_{\sigma},$$

where $(\Phi_n)_{ij} = (-1)^{i-1}\delta_{i,n-j+1}$.

This map is a special case of endoscopic functorality as $U(n_1) \times \cdots \times U(n_r)$ is not a Levi subgroup of $U(n)$ if $r > 1$. Also the map ξ depends on the choice of character μ , it is unique if one fixes a character μ . From now onwards, we fix a character μ once and for all.

3.2 Endoscopic Transfer for unramified representation of $U(n)$

Endoscopic transfer is a special case of Langland's functoriality. If $n = n_1 + \cdots + n_r$, then $H = U(n_1) \times \cdots \times U(n_r)$ is an endoscopic subgroup of $U(n)$. The map ξ defined as in (4), defines the map of corresponding L-groups. Using ξ , we transfer parameters of H to that of $U(n)$, we call it endoscopic transfer.

Let us now restrict to the unramified parameter. We start with an unramified parameter ψ_l for H . So the parameter ψ_l is trivial on the the inertia group, hence factor through \mathbb{Z} , and is determined by the image of Frobenius Φ . We compose this with the map ξ , to get an unramified parameter for $U(n)$.

$$\begin{array}{ccccc} W_{\mathbb{Q}_l} \times SL_2(\mathbb{C}) & \xrightarrow{\psi_l} & {}^L U(n_1) \times \cdots \times U(n_r) & \xrightarrow{\xi} & {}^L U(n) \\ \downarrow & & \nearrow \psi_l & & \\ W_{\mathbb{Q}_l}/I_{\mathbb{Q}_l} & \xrightarrow{\cong} & \mathbb{Z} & & \end{array} \quad (6)$$

Since unramified parameters are in bijection with unramified representations[3, Proposition 1.12.1], we get endoscopic transfer for unramified representation of $U(n)$.

3.3 Global Endoscopic Transfer for $U(n)$

Let F be a number field and L_F denotes the conjectural Langlands group[10].

Lemma 3.1. [12, Lemma 2.2.1] Let ψ be a discrete parameter for $U(n)$, that is a morphism $\psi : L_{\mathbb{Q}} \times SL_2(\mathbb{C}) \rightarrow {}^L U(n)$. Then $\psi_E : L_E \times SL_2(\mathbb{C}) \rightarrow GL_m(\mathbb{C})$ is a sum of pairwise nonisomorphic irreducible representation of ρ_j such that $\rho_j^\perp \cong \rho_j$.

This lemma allows us to define stable A-parameter.

Definition 3.2. We call a discrete A-parameter ψ stable if the base change to ψ_E is irreducible. An A-parameter which is not stable is called endoscopic.

Let $n = n_1 + \cdots + n_r$ be a partition and for $j = 1, \dots, r$, let ψ_j be a parameter of $U(n_j)$ of the form $\psi_j(\gamma \times h) = \alpha_j(\gamma \times h) \times w_\gamma$. We denote the product parameter $\psi_1 \times \cdots \times \psi_r$ for $U(n_1) \times \cdots \times U(n_r)$ by

$$\gamma \times h \rightarrow (\alpha_1(\gamma \times h) \times \cdots \times \alpha_r(\gamma \times h)) \times w_\gamma.$$

The equivalence class of $\psi_1 \times \cdots \times \psi_r$ is independent of the ordering of ψ_j .

Following lemma describes endoscopic parameter for $U(n)$.

Lemma 3.3. [12, Lemma 2.2.2] Let ψ be a discrete parameter for $U(n)$. Then there is a unique (unordered) partition $n = n_1 + \cdots + n_r$ and distinct stable parameters ψ_j for $U(n_j)$ such that $\psi = \xi \circ (\psi_1 \times \cdots \times \psi_r)$.

Conversely, if $n = n_1 + \cdots + n_r$ and ψ_j is a stable parameter for $U(n_j)$, then $\xi \circ (\psi_1 \times \cdots \times \psi_r)$ is a discrete parameter for $U(n)$ if and only if the ψ_j are distinct. Here ξ is the map defined in 4.

In view of previous lemma, a discrete parameter of $U(n)$ can be uniquely as $\psi = \xi \circ (\psi_1 \times \cdots \times \psi_r)$, where ψ_j are distinct stable parameter for $U(n_j)$. So a discrete parameter of $U(n)$ is endoscopic if $r > 1$.

We make the following assumptions regarding A-packets of $U(n)$.

- A packets are defined for $U(n)$.
- Π_1, Π_2 discrete A-packets such that for almost all v , $\Pi_{1,v}$ and $\Pi_{2,v}$ contain the same unramified representations, then Π_1 and Π_2 coincide.

Definition 3.4. Let $\Pi = \bigotimes_v \Pi_v$ be a discrete A-packet for $U(n)$. Then Π is stable if there exists a discrete representation $\pi = \bigotimes_v \pi_v$ of GL_n/E such that for all v for which Π_v is unramified, the base change $(\Pi_v)_E$ coincide with π_v (For unramified representation, base change map is well defined). If π is cuspidal, we call Π cuspidal. An A-packet which is not stable is called Endoscopic.

Arthur's conjecture predicts existence of a natural correspondence which associates to every global discrete A-parameter of $U(n)$ (upto equivalence) an A-packet of $U(n)$, or the empty set. Under the conjecture stable(resp. endoscopic) A-parameter of $U(n)$ corresponds to a stable(resp. endoscopic) A-packet of $U(n)$. Predictions of Arthur's conjecture for $U(n)$ were verified by Rogawski when $n \leq 3$ [11]. Main obstruction to prove the existence of global Endoscopic transfer for $U(n)$ and arthur conjecture for $U(n)$ for $n > 3$ was the proof of fundamental lemma. Since by the work of Ngô, fundamental lemma is proved, a group of mathematicians are working on writing down the details of endoscopic transfer of unitary group and verification of Arthur's conjecture.

We will assume that the global endoscopic transfer for $U(n)$ exists, in fact one can assume that the global endoscopic transfer for $U(n)$ exists for almost all automorphic representations. This is expected to be true by the work of Closel, Harris, Labesse, Ngô, and many other mathematicians (see [7], volume 2 under preparation) .

4 Maps between eigenvarieties of $U(n)$ interpolating endoscopic transfer

Let \mathcal{E}_n , \mathcal{E}_{n_1} and \mathcal{E}_{n_2} denote the eigenvariety associated to $U(n)$, $U(n_1)$ and $U(n_2)$ respectively, where $n = n_1 + n_2$. In this section we will construct a map $f : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n$, which at classical points interpolates endoscopic transfer. But first we prove two simple lemmas regarding system of eigenvalues, which will be used during the construction.

Lemma 4.1. *Let M be a finite dimensional vector space over $\bar{\mathbb{Q}}_p$. Let $H \subset H'$ be two commutative $\bar{\mathbb{Q}}_p$ algebras acting on M . Then any H system of eigenvalue appearing in M , extends to a H' system of eigenvalue appearing in M .*

Proof. Let χ be a H system of eigenvalue appearing in M . Let $M[\chi]$ be the corresponding eigenspace. Let $m_1 \in M[\chi]$, $h \in H$, $h' \in H'$, then

$$h.(h'.m_1) = h'.(h.m_1) = h'.(\chi(h)m_1) = \chi(h)(h'.m_1)$$

So h acts on $h'.m_1$ by $\chi(h)$. Hence $h'.m_1 \in M[\chi]$. So $M[\chi]$ is stable under the action of H' . Since for any commutative algebra acting on a finite dimensional vector space over an algebraically closed field has a common eigenvector, we see there exists a $m \in M[\chi]$, a common eigenvector for the action of H' on $M[\chi]$. Hence m is an eigenvector for the action of H' on M . So χ extends to a H' system of eigenvalue in M . \square

Note: A system of H' eigenvalue which extends a given system of H eigenvalue need not be unique.

Lemma 4.2. *Let H_1, H_2 be two $\bar{\mathbb{Q}}_p$ algebras acting semisimply on a finite dimensional $\bar{\mathbb{Q}}_p$ vector space M_1, M_2 via ψ_1, ψ_2 respectively. Then a system of $H_1 \otimes H_2$ eigenvalue appearing in $M_1 \otimes M_2$ is same as a system of H_1 eigenvalue appearing in M_1 and a system of H_2 eigenvalue appearing in M_2 composed via tensor product.*

Proof. Let χ_1 be a system of H_1 eigenvalue for M_1 and χ_2 be a system of H_2 eigenvalue for M_2 , that is, there exists $v_1 \in M_1$, such that, $\chi_1 : H_1 \rightarrow \bar{\mathbb{Q}}_p, \chi_1(h_1)v_1 = \psi_1(h_1)v_1$ for all $h_1 \in H_1$, and there exists $v_2 \in M_2$, such that, $\chi_2 : H_2 \rightarrow \bar{\mathbb{Q}}_p, \chi_2(h_2)v_2 = \psi_2(h_2)v_2$ for all $h_2 \in H_2$.

We want to show that, $\chi = \chi_1 \otimes \chi_2 : H_1 \otimes H_2 \rightarrow \bar{\mathbb{Q}}_p$ defined by $\chi(h_1 \otimes h_2) = \chi_1(h_1)\chi_2(h_2)$ is a system of eigenvalue for $H_1 \otimes H_2$ appearing in $M_1 \otimes M_2$. Let us define $\psi = \psi_1 \otimes \psi_2$. Let $h = \sum_i \alpha_i(h_{1,i} \otimes h_{2,i})$ be an element of $H_1 \otimes H_2$. Then we have,

$$\begin{aligned} \psi(h)(v_1 \otimes v_2) &= \sum_i \alpha_i \psi(h_{1,i} \otimes h_{2,i})(v_1 \otimes v_2) \\ &= \sum_i \alpha_i \psi_1(h_{1,i})v_1 \otimes \psi_2(h_{2,i})v_2 \\ &= \sum_i \alpha_i \chi_1(h_{1,i})v_1 \otimes \chi_2(h_{2,i})v_2 \\ &= \chi\left(\sum_i \alpha_i(h_{1,i} \otimes h_{2,i})\right)(v_1 \otimes v_2) \\ &= \chi(h)(v_1 \otimes v_2). \end{aligned}$$

Hence χ is a system of eigenvalue for $H_1 \otimes H_2$ appearing in $M_1 \otimes M_2$.

Let χ be a system of eigenvalue for $H_1 \otimes H_2$ appearing in $M_1 \otimes M_2$. We want show that $\chi = \chi_1 \otimes \chi_2$ for some χ_1 (resp. χ_2) a system of eigenvalue for H_1 (resp. H_2) appearing in M_1 (resp. M_2). Since H_i acts semi-simply on M_i , we have,

$$M_1 \cong \oplus_{i=1}^r M_1[\chi_{1,i}],$$

$$M_2 \cong \oplus_{j=1}^s M_2[\chi_{2,j}].$$

Let $\dim M_1[\chi_{1,i}] = n_i$ generated by $\{a_1, \dots, a_{n_i}\}$, and $\dim M_2[\chi_{2,j}] = m_j$ generated by $\{b_1, \dots, b_{m_j}\}$. Then we have,

$$\dim M_1[\chi_{1,i}] \otimes M_2[\chi_{2,j}] = n_i m_j = (\dim M_1[\chi_{1,i}])(\dim M_2[\chi_{2,j}]),$$

generated by $\{a_k \otimes b_l\}_{k=1, \dots, n_i; l=1, \dots, m_j}$. By previous part, $\chi_{1,i} \otimes \chi_{2,j}$ is a system of $H_1 \otimes H_2$ eigenvalue for $M_1 \otimes M_2$ with $a_k \otimes b_l$ as an eigenvector. Hence we have, $M_1 \otimes M_2[\chi_{1,i} \otimes \chi_{2,j}] \supseteq M_1[\chi_{1,i}] \otimes M_2[\chi_{2,j}]$. We see that,

$$\begin{aligned} \dim M_1 \otimes M_2[\chi_{1,i} \otimes \chi_{2,j}] &\geq \dim M_1[\chi_{1,i}] \otimes M_2[\chi_{2,j}], \\ &= (\dim M_1[\chi_{1,i}])(\dim M_2[\chi_{2,j}]). \end{aligned}$$

Summing over all i and j , we get,

$$\begin{aligned} \sum_{i=1}^r \sum_{j=1}^s \dim M_1 \otimes M_2[\chi_{1,i} \otimes \chi_{2,j}] &\geq \sum_{i=1}^r \sum_{j=1}^s (\dim M_1[\chi_{1,i}])(\dim M_2[\chi_{2,j}]), \\ &= (\dim M_1)(\dim M_2), \\ &= \dim(M_1 \otimes M_2). \end{aligned}$$

Thus we have,

$$M_1 \otimes M_2 \cong \oplus_{i,j} M_1 \otimes M_2[\chi_{1,i} \otimes \chi_{2,j}].$$

Hence any system of eigenvalue χ of $H_1 \otimes H_2$ in $M_1 \otimes M_2$ is of the form $\chi_{1,i} \otimes \chi_{2,j}$. \square

A key ingredient to construct the map is a version of comparison theorem due to Chenevier[6], which we recall below.

Let us fix an eigenvariety data, a ring \mathcal{H} with a distinguished element U_p , \mathcal{W} a reduced rigid space with an admissible covering \mathfrak{C} , and Banach modules M_W with an action of \mathcal{H} for all $W \in \mathfrak{C}$, which satisfies compatibility criterion.

Definition 4.3. [1, Definition II.5.4] A classical structure on eigenvariety data is the data of

(CSD1) a very Zariski dense subset $X \subset \mathcal{W}$,

(CSD2) for every $x \in X$, a finite dimensional \mathcal{H} module M_x^{cl}

such that

(CSC1) for every $x \in X$, there exists an \mathcal{H} equivariant injective map $M_x^{cl,ss} \hookrightarrow M_x^{fs,ss}$

(CSC2) for every $\nu \in \mathbb{R}$, let X_ν be the set of $x \in X$, such that there exists an \mathcal{H} -equivariant isomorphism $M_x^{cl, \leq \nu} \cong M_x^{fs, \leq \nu}$, then for every $x \in X$, there exists a basis of neighborhoods of $x \in V$, such that $X_\nu \cap V$ is Zariski dense in V .

Theorem 4.4. *Comparison Theorem(Chenevier) [1, Theorem II.5.6] Suppose that we have two eigenvariety data with same $\mathcal{H}, \mathcal{W}, \mathfrak{C}$, but different Banach modules M_W and M'_W . Let us call \mathcal{E} and \mathcal{E}' the two eigenvarieties attached to those data. Assume that the two eigenvarieties are each*

provided with a classical structure with the same set X (CSD1), but different \mathcal{H} modules M_x^{cl} and $M_x'^{\text{cl}}$. Suppose that for every $x \in X$, there exists an \mathcal{H} -equivariant injective map

$$M_x'^{\text{cl},ss} \hookrightarrow M_x^{\text{cl},ss}.$$

Then there exists a unique closed embedding $\mathcal{E}' \hookrightarrow \mathcal{E}$ of the eigenvarieties compatible with weight maps to \mathcal{W} and with the maps $\mathcal{H} \rightarrow \mathcal{O}(\mathcal{E})$ and $\mathcal{H} \rightarrow \mathcal{O}(\mathcal{E}')$.

Let $u_i = \text{diag}(1, \dots, 1, p, 1, \dots, 1)$, where p occurs at i -th place. Let $F_i = \psi(u_i)$, where $\psi : \mathcal{H} \rightarrow \mathcal{O}(\mathcal{E}_n)$. Let $\mathcal{Z}(n)$ be the subset of $x \in \mathcal{E}_n(\bar{\mathbb{Q}}_p)$ such that,

- (a) $\omega(x) = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^{n,-}$,
- (b) $v(F_1(x)F_2(x) \cdots F_{n-1}(x)) < 1 + \min_{i=1}^{n-1} (k_i - k_{i+1})$,
- (c) if $\phi_i := F_i(x)p^{-k_i+i-1}$, then for all $i \neq j$, $\phi_i \phi_j^{-1} \neq p$.

Then $\mathcal{Z}(n)$ is a Zariski accumulation dense subset of \mathcal{E}_n [2, Theorem 7.3.1]. Let $\mathcal{Z}_{\text{reg}}(n)$ be the subset of $\mathcal{Z}(n)$, parameterizing the p -refined (π, \mathcal{R}) such that π_∞ is regular, and such that the semisimple conjugacy class of π_p has n distinct eigenvalues. Then $\mathcal{Z}_{\text{reg}}(n)$ is a Zariski dense subset of \mathcal{E}_n accumulating at each point of $\mathcal{Z}(n)$ [2, Lemma 7.5.3].

Let $\kappa = (\kappa_1, \dots, \kappa_n) : \mathcal{E}_n \rightarrow \mathbb{A}^n$ is the composition of the map $\log_p \circ \omega$ by the affine change of co-ordinates $(x_1, \dots, x_n) \mapsto (-x_1, -x_2 + 1, \dots, -x_n + n - 1)$. Then for $z = (\pi, \mathcal{R}) \in \mathcal{Z}_{\text{reg}}$, with $\omega(z) = (k_1, k_2, \dots, k_n)$, we have $\kappa_i(z) = -k_i + i - 1$, and $\kappa_1(z), \dots, \kappa_n(z)$ is the strictly increasing sequence of Hodge-Tate weights of ρ_π , where ρ_π is the Galois representation associated to π [2, Definition 7.5.11]. Moreover, we have [2, Definition 7.2.13],

$$i_p i_\infty^{-1}(\mathcal{R}|p|^{\frac{1-n}{2}}) = (F_1(z)p^{\kappa_1(z)}, \dots, F_i(z)p^{\kappa_i(z)}, \dots, F_n(z)p^{\kappa_n(z)}).$$

Let $z_1 \in \mathcal{Z}_{\text{reg}}(n_1)$ be any point with $\omega(z_1) = (k_1, \dots, k_{n_1}) \in \mathbb{Z}^{n_1,-}$, and $z_2 \in \mathcal{Z}_{\text{reg}}(n_2)$ be any point with $\omega(z_2) = (k'_1, \dots, k'_{n_2}) \in \mathbb{Z}^{n_2,-}$. We call the pair $(z_1, z_2) \in \mathcal{Z}_{\text{reg}}(n_1) \times \mathcal{Z}_{\text{reg}}(n_2)$ a “nice” classical point if we have $k_{n_1} > k'_1$.

Theorem 4.5. Endoscopic Transfer on Eigenvariety for $U(n)$ *There exists a unique map of eigenvarieties*

$$f : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n,$$

where $n = n_1 + n_2$, such that a “nice” classical point of $\mathcal{E}_{n_1} \times \mathcal{E}_{n_2}$, $((\pi_1, \mathcal{R}_1), (\pi_2, \mathcal{R}_2))$ maps to a classical point of \mathcal{E}_n , (π, \mathcal{R}) , where π is the endoscopic transfer of π_1 and π_2 and $\mathcal{R} = (\mu_1(\text{Frob}_p)\mathcal{R}_1, \mu_2(\text{Frob}_p)\mathcal{R}_2)$, here μ_1 and μ_2 as in (5).

Proof. (Uniqueness) Let $\phi', \phi'' : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n$ be two such maps interpolating endoscopic transfer at “nice” classical points. Let $z \in \mathcal{E}_{n_1} \times \mathcal{E}_{n_2}$ be a “nice” classical point, that is $z = ((\pi_1, \mathcal{R}_1), (\pi_2, \mathcal{R}_2))$, where π_i is a p -refined automorphic form for $U(n_i)$ of weight \underline{k}_i , where $\underline{k}_1 = (k_1, \dots, k_{n_1})$ and $\underline{k}_2 = (k_{n_1+1}, \dots, k_n)$ with $k_1 > k_2 > \dots > k_n$. Then π , the endoscopic transfer of π_1 and π_2 has weight $\underline{k} = (k_1, k_2, \dots, k_n)$. Suppose $\phi'(z) = (\pi, \mathcal{R}')$ and $\phi''(z) = (\pi, \mathcal{R}'')$, where $\mathcal{R}', \mathcal{R}''$ are any two accessible refinements of π_p . Let $\mathcal{R}'' = \sigma_z(\mathcal{R}')$, where $\sigma_z \in S_n$, as \mathcal{R}'' is an ordering of \mathcal{R}' . The set of “nice” classical points can be written as,

$$Z = \bigsqcup_{\sigma \in S_n} \{z \in Z | \sigma_z = \sigma\} := \bigsqcup_{\sigma \in S_n} Z_\sigma.$$

Since Z is dense, at least one of the Z_σ is dense. Suppose Z_{σ_0} is dense, we will use $z \in Z_{\sigma_0}$, hence $\sigma_0(\mathcal{R}') = \mathcal{R}''$, for all $z \in Z_{\sigma_0}$.

Note that,

$$i_p i_\infty^{-1}(\mathcal{R}'|p|^{\frac{1-n}{2}}) = (F_1(\phi'(z))p^{\kappa_1(\phi'(z))}, \dots, F_i(\phi'(z))p^{\kappa_i(\phi'(z))}, \dots, F_n(\phi'(z))p^{\kappa_n(\phi'(z))}),$$

and

$$i_p i_\infty^{-1}(\mathcal{R}''|p|^{\frac{1-n}{2}}) = (F_1(\phi''(z))p^{\kappa_1(\phi''(z))}, \dots, F_i(\phi''(z))p^{\kappa_i(\phi''(z))}, \dots, F_n(\phi''(z))p^{\kappa_n(\phi''(z))}).$$

Thus we have,

$$F_{\sigma_0(i)}(\phi'(z))p^{\kappa_{\sigma_0(i)}(\phi'(z))} = F_i(\phi''(z))p^{\kappa_i(\phi''(z))},$$

and hence we have,

$$F_i(\phi''(z)) = F_{\sigma_0(i)}(\phi'(z))p^{\kappa_{\sigma_0(i)}(\phi'(z)) - \kappa_i(\phi''(z))}.$$

Since $\kappa_i(\phi'(z)) = \kappa_i(\phi''(z))$ for all i , as κ_i is the strictly increasing sequence of Hodge-Tate weights of the Galois representation ρ_π , it is independent of the choice of the refinement. We have,

$$F_i(\phi''(z)) = F_{\sigma_0(i)}(\phi'(z))p^{\kappa_{\sigma_0(i)}(\phi'(z)) - \kappa_i(\phi'(z))},$$

and hence,

$$F_i(\phi''(z)) = F_{\sigma_0(i)}(\phi'(z))p^{(\kappa_{\sigma_0(i)} - \kappa_i)(\phi'(z))}.$$

The left hand side of the equation is an analytic function, but the right hand side is not analytic unless $\sigma_0(i) = i$ for all i . Thus $\mathcal{R}' = \mathcal{R}''$. Hence $\phi'(z) = \phi''(z)$ for all $z \in Z_{\sigma_0}$. Since Z_{σ_0} is dense, we get $\phi' = \phi''$.

Existance: Let \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H} be Hecke algebras associated to $U(n_1)$, $U(n_2)$ and $U(n)$ respectively. We have, $\mathcal{H}_1 \cong \otimes_{l \in S_0} \mathcal{H}_{1,l} \otimes \mathcal{A}_{p,1}$, where $\mathcal{H}_{1,l} \cong \bar{\mathbb{Q}}_p[X_{1,l}, \dots, X_{n_1,l}, X_{1,l}^{-1}, \dots, X_{n_1,l}^{-1}]^{S_{n_1}}$ and $\mathcal{A}_{p,1} \cong \bar{\mathbb{Q}}_p[X_1, \dots, X_{n_1}, X_1^{-1}, \dots, X_{n_1}^{-1}]$. Similarly, we have, $\mathcal{H}_2 \cong \otimes_{l \in S_0} \mathcal{H}_{2,l} \otimes \mathcal{A}_{p,2}$, where $\mathcal{H}_{2,l} \cong \bar{\mathbb{Q}}_p[Y_{1,l}, \dots, Y_{n_2,l}, Y_{1,l}^{-1}, \dots, Y_{n_2,l}^{-1}]^{S_{n_2}}$ and $\mathcal{A}_{p,2} \cong \bar{\mathbb{Q}}_p[Y_1, \dots, Y_{n_2}, Y_1^{-1}, \dots, Y_{n_2}^{-1}]$ and $\mathcal{H} \cong \otimes_{l \in S_0} \mathcal{H}_l \otimes \mathcal{A}_p$, where $\mathcal{H}_l \cong \bar{\mathbb{Q}}_p[T_{1,l}, \dots, T_{n,l}, T_{1,l}^{-1}, \dots, T_{n,l}^{-1}]^{S_n}$ and $\mathcal{A}_p \cong \bar{\mathbb{Q}}_p[T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}]$.

We have an injection of \mathcal{H}_l into $\mathcal{H}_{1,l} \otimes \mathcal{H}_{2,l}$ given by,

$$\theta : \mathcal{H}_l \hookrightarrow \mathcal{H}_{1,l} \otimes \mathcal{H}_{2,l} \tag{7}$$

$$\theta(T_{i,l}) = \begin{cases} \mu_1(Frob_l)X_{i,l} & \text{if } 1 \leq i \leq n_1, \\ \mu_2(Frob_l)Y_{i-n_1,l} & \text{if } n_1 < i \leq n, \end{cases}$$

where μ_i as in (5), and $Frob_l$ is the frobenius.

Also we have an isomorphism of Atkin-Lehner algebra given by,

$$\mathcal{A}_p \cong \mathcal{A}_{p,1} \otimes \mathcal{A}_{p,2} \tag{8}$$

$$T_i \mapsto \begin{cases} \mu_1(Frob_p)p^{\frac{n_2}{2}}X_i & \text{if } 1 \leq i \leq n_1, \\ \mu_2(Frob_p)p^{\frac{-n_1}{2}}Y_{i-n_1} & \text{if } n_1 < i \leq n. \end{cases}$$

Combining these two maps, we get an injection $G : \mathcal{H} \hookrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$.

Notice that $\mathcal{W}_{n_1} \times \mathcal{W}_{n_2} \cong \mathcal{W}_n$. Let $W \subset \mathcal{W}_n$ be an affinoid open such that $W \cong W_1 \times W_2$, and $W_i \subset \mathcal{W}_{n_i}$ is an affinoid open. Let R be the affinoid algebra so that $\text{Spec}(R) = W$. Let $M_W = \mathcal{S}(W, r)$ and $M'_W = M_{W_1} \otimes M_{W_2} = \mathcal{S}(W_1, r) \otimes \mathcal{S}(W_2, r)$, with $r \geq \max(r_W, r_{W_1}, r_{W_2})$. Then M_W is a \mathcal{H} module and M'_W is a $\mathcal{H}_1 \otimes \mathcal{H}_2$ module and hence a \mathcal{H} module. Using (R, M_W, \mathcal{H}) , we construct the local piece of eigenvariety \mathcal{E}_n , using $(R, M'_W, \mathcal{H}_1 \otimes \mathcal{H}_2)$ we construct local piece of eigenvariety $\mathcal{E}_{n_1} \times \mathcal{E}_{n_2}$. Let us denote by \mathcal{E}' the eigenvariety whose local pieces are constructed

from the data (R, M'_W, \mathcal{H}) . Since $\mathcal{H} \hookrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$, there is a surjective map $\alpha : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}'$. So to construct a map $f : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n$, we need to construct a map $\beta : \mathcal{E}' \rightarrow \mathcal{E}_n$.

To construct the map β , we use the comparison theorem due to Chenevier (4.4). We first need to define a classical structure on \mathcal{E} and \mathcal{E}' . For $x \in \mathcal{W}(L)$, we shall denote by $(M_x)^{fs}$ (resp. $(M'_x)^{fs}$), the finite slope part of the fiber at x of the eigenvariety data used to construct \mathcal{E} (resp. \mathcal{E}'), base changed to $\bar{\mathbb{Q}}_p$. So if we have $x = \underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^{n,-}$, then, $(M_{\underline{k}})^{fs} = \mathcal{S}(\underline{k}, r)^{fs} \otimes \bar{\mathbb{Q}}_p$ and $(M'_{\underline{k}})^{fs} = (\mathcal{S}(\underline{k}_1, r)^{fs} \otimes \bar{\mathbb{Q}}_p) \otimes (\mathcal{S}(\underline{k}_2, r)^{fs} \otimes \bar{\mathbb{Q}}_p)$, where $\underline{k}_1 = (k_1, \dots, k_{n_1})$ and $\underline{k}_2 = (k_{n_1+1}, \dots, k_n)$. In both cases take $X = \mathbb{Z}^{n,-} \subset \mathcal{W}$ (CSD1). For $x = \underline{k} = (\underline{k}_1, \underline{k}_2) = (k_1, \dots, k_n) \in \mathbb{Z}^{n,-}$, define $(M_{\underline{k}})^{cl} = (F(W_{\underline{k}}(L)^*)^{fs} \otimes \delta_{\underline{k}}) \otimes \bar{\mathbb{Q}}_p$, $(M'_{\underline{k}})^{cl} = ((F(W_{\underline{k}_1}(L)^*)^{fs} \otimes \delta_{\underline{k}_1}) \otimes \bar{\mathbb{Q}}_p) \otimes ((F(W_{\underline{k}_2}(L)^*)^{fs} \otimes \delta_{\underline{k}_2}) \otimes \bar{\mathbb{Q}}_p)$. Since we have natural \mathcal{H} -equivariant embedding $F(W_{\underline{k}}(L) \otimes \delta_{\underline{k}}) \hookrightarrow \mathcal{S}(\underline{k}, r)^{fs}$, we have as \mathcal{H} module $(M_{\underline{k}})^{cl}$ (resp. $(M'_{\underline{k}})^{cl}$), are submodule of $(M_{\underline{k}})^{fs}$ (resp. $(M'_{\underline{k}})^{fs}$), hence condition (CSC1) is satisfied. For condition (CSC2), fix $\nu \in \mathbb{R}$. Define,

$$X_\nu = \{\underline{k} \in \mathbb{Z}^{n,-} | k_i - k_{i+1} + 1 > \nu \text{ for all } i = 1, \dots, (n-1)\}.$$

Thus by Chenevier's control theorem (2.1) we have, for any $\underline{k} \in X_\nu$, $(M_{\underline{k}})^{cl, \leq \nu} \simeq (M_{\underline{k}})^{fs, \leq \nu}$ and $(M'_{\underline{k}})^{cl, \leq \nu} \simeq (M'_{\underline{k}})^{fs, \leq \nu}$. Hence the condition (CSC2) is satisfied for all cases. Having defined classical structure on \mathcal{E} and \mathcal{E}' , we need to construct an \mathcal{H} -equivariant injective map $(M'_x)^{cl, ss} \hookrightarrow (M_x)^{cl, ss}$ for all $x \in X$, to get the map $\beta : \mathcal{E}' \hookrightarrow \mathcal{E}$. Thus we want to show, every \mathcal{H} system of eigenvalue appearing in $(M'_x)^{cl, ss}$ also appears as an \mathcal{H} system of eigenvalue in $(M_x)^{cl, ss}$.

Let χ be a \mathcal{H} system of eigenvalue appearing in $(M'_{\underline{k}})^{cl, ss}$. By lemma 4.1 we can lift χ to $\tilde{\chi}$ a $\mathcal{H}_1 \otimes \mathcal{H}_2$ system of eigenvalue in $(M'_{\underline{k}_1})^{cl, ss}$. Then by lemma 4.2 $\tilde{\chi} = \chi_1 \otimes \chi_2$, where χ_i is a \mathcal{H}_i system of eigenvalue appearing in $((F(W_{\underline{k}_1}(L)^*)^{fs} \otimes \delta_{\underline{k}_1}) \otimes \bar{\mathbb{Q}}_p)^{ss}$. So χ_i corresponds to (π, \mathcal{R}_i) , where π_i is an automorphic representation of $U(n_i)$ of weight \underline{k}_i and \mathcal{R}_i is an accessible refinement of $\pi_{i,p}$. Let π be the endoscopic transfer of π_1 and π_2 . Let $\mathcal{R} = (\mu_1(Frob_p)\mathcal{R}_1, \mu_2(Frob_p)\mathcal{R}_2)$. Let χ' be the \mathcal{H} system of eigenvalue in $(M_{\underline{k}})^{cl, ss}$ corresponding to (π, \mathcal{R}) . We want to show $\chi' = \chi$. To prove this it is enough to show $\chi'_p = \chi_p$ and $\chi'_l = \chi_l$ for $l \neq p$.

Lemma 4.6. $\chi'_l = \chi_l$, where χ_l and χ'_l are as described above.

Proof. Let $\pi_{i,l}$ be the unramified representation of GL_{n_i} corresponding to $\chi_{i,l}$. Since $\pi_{1,l}$ and $\pi_{2,l}$ are unramified representations of GL_{n_1} and GL_{n_2} respectively, they are determined by their Satake parameters say $(\lambda_{1,l}, \dots, \lambda_{n_1,l})$ and $(\lambda'_{1,l}, \dots, \lambda'_{n_2,l})$ respectively.

Under the endoscopic transfer map, $\pi_{1,l}$ and $\pi_{2,l}$ maps to a unramified representation of GL_n as follows:

$$\mathbb{Z} \longrightarrow LU(n_1) \times U(n_2) \xrightarrow{\xi} LU(n)$$

$Frob_l \longmapsto (diag(\lambda_{1,l}, \dots, \lambda_{n_1,l}), diag(\lambda'_{1,l}, \dots, \lambda'_{n_2,l})) \times Frob_l \longmapsto diag(\lambda_{1,l}, \dots, \lambda_{n_1,l}, \lambda'_{1,l}, \dots, \lambda'_{n_2,l}) \xi(Frob_l)$
Since $\xi(Frob_l) = \xi(\mu_1(Frob_l)I_{n_1}, \mu_2(Frob_l)I_{n_2}) \times Frob_l$, we have under endoscopic transfer as in (6),

$$Frob_l \mapsto diag(\mu_1(Frob_l)\lambda_1, \dots, \mu_1(Frob_l)\lambda_{n_1}, \mu_2(Frob_l)\lambda'_1, \dots, \mu_2(Frob_l)\lambda'_{n_2}) \times Frob_l.$$

Hence if π_l is the endoscopic transfer of $\pi_{1,l}$ and $\pi_{2,l}$, then the Satake parameter for π_l is given by $(\mu_1(Frob_l)\lambda_{1,l}, \dots, \mu_1(Frob_l)\lambda_{n_1,l}, \mu_2(Frob_l)\lambda'_{1,l}, \dots, \mu_2(Frob_l)\lambda'_{n_2,l})$. We want to show π_l corresponds to χ_l in M_x . Since Satake parameter of $\pi_{1,l}$ is $(\lambda_{1,l}, \dots, \lambda_{n_1,l})$, which corresponds to $\chi_{1,l}$,

we have,

$$\begin{aligned}\chi_{1,l} : H_{1,l} &\rightarrow \bar{\mathbb{Q}}_p \\ \chi_{1,l}(s_{k,1,l}) &= \sum_{1 \leq i_1 < \dots < i_k \leq n_1} \lambda_{i_1,l} \cdots \lambda_{i_k,l},\end{aligned}$$

where $s_{k,1}$ is symmetric polynomial in $X_{1,l}, \dots, X_{n_1,l}$ of degree k . Similarly we have,

$$\chi_{2,l}(s_{k,2,l}) = \sum_{1 \leq j_1 < \dots < j_k \leq n_2} \lambda'_{j_1,l} \cdots \lambda'_{j_k,l},$$

where $s_{k,2,l}$ is symmetric polynomial in $Y_{1,l}, \dots, Y_{n_2,l}$ of degree k .

By the work of Satake, $\bar{\mathbb{Q}}_p[X_1, \dots, X_{n_1}, X_1^{-1}, \dots, X_{n_1}^{-1}]$ is integral over $\bar{\mathbb{Q}}_p[X_1, \dots, X_{n_1}, X_1^{-1}, \dots, X_{n_1}^{-1}]^{S_{n_1}}$, and any character of $\bar{\mathbb{Q}}_p[X_1, \dots, X_{n_1}, X_1^{-1}, \dots, X_{n_1}^{-1}]^{S_{n_1}}$ can be lifted to a character of the ring $\bar{\mathbb{Q}}_p[X_1, \dots, X_{n_1}, X_1^{-1}, \dots, X_{n_1}^{-1}]$. Hence, $\chi_{1,l}$ and $\chi_{2,l}$ can be extended to character $\tilde{\chi}_{1,l}$ and $\tilde{\chi}_{2,l}$ of $\bar{\mathbb{Q}}_p[X_1, \dots, X_{n_1}, X_1^{-1}, \dots, X_{n_1}^{-1}]$ and $\bar{\mathbb{Q}}_p[Y_1, \dots, Y_{n_2}, Y_1^{-1}, \dots, Y_{n_2}^{-1}]$ respectively.

Let us assume $\tilde{\chi}_{1,l}(X_{i,l}) = \lambda_{i,l}$ and $\tilde{\chi}_{2,l}(Y_{j,l}) = \lambda'_{j,l}$.

Let $\tilde{\chi}_l = \tilde{\chi}_{1,l} \otimes \tilde{\chi}_{2,l}$, then

$$\tilde{\chi}_l(T_{i,l}) = \begin{cases} \tilde{\chi}_{1,l}(\mu_1(Frob_l)X_{i,l}) = \mu_1(Frob_l)\lambda_{i,l} & \text{if } 1 \leq i \leq n_1, \\ \tilde{\chi}_{2,l}(\mu_2(Frob_l)Y_{i-n_1,l}) = \mu_2(Frob_l)\lambda'_{i-n_1,l} & \text{if } n_1 < i \leq n. \end{cases}$$

Note that $\tilde{\chi}_l|_{\mathcal{H}_{1,l} \otimes \mathcal{H}_{2,l}} = \chi_{1,l} \otimes \chi_{2,l}$ and $\tilde{\chi}_l|_H = \chi_l$. Hence χ_l corresponds to π_l , the endoscopic transfer of $\pi_{1,l}$ and $\pi_{2,l}$. This is independent of the choice of $\tilde{\chi}_{1,l}$ and $\tilde{\chi}_{2,l}$ as even if we choose different lifts, the values of $X_{i,l}, Y_{j,l}, s_{k,1,l}, s_{k,2,l}$ will be different, but the values of $s_{k,l}$, the symmetric polynomial in $T_{1,l}, \dots, T_{n,l}$, will be same. And hence the Satake parameter corresponding to χ_l will be same. Since by definition of χ', χ'_l corresponds to π_l , we have $\chi'_l = \chi_l$. \square

Lemma 4.7. $\chi'_p = \chi_p$, where χ_p and χ'_p are defined as above.

Proof. By the lemma 4.2, every system of \mathcal{A}_p eigenvalue χ_p in $(M'_{\underline{k}})^{cl,ss}$, is of the form $\chi_{p,1} \otimes \chi_{p,2}$, where $\chi_{p,i}$ is an $\mathcal{A}_{p,i}$ system of eigenvalue appearing in $(M'_{\underline{k}_i})^{cl,ss}$. So $\chi_{p,i}$ corresponds to $(\pi_{p,i}, \mathcal{R}_i)$, where $\pi_{p,i}$ is an unramified representation of GL_{n_i} at p , and \mathcal{R}_i is an accessible refinement. Then as in equation (2), we have

$$\chi_{p,i}|_U = \gamma_i \delta_{B_i}^{-1/2} \delta_{\underline{k}_i}, \quad (9)$$

where B_i is the borel subgroup of GL_{n_i} , δ_{B_i} is the modulus character, $\gamma_i : U_i \rightarrow \mathcal{C}^*$ corresponds to the refinement \mathcal{R}_i and $\delta_{\underline{k}_i}$ as in (2).

Note:1 Since $\underline{k} = (\underline{k}_1, \underline{k}_2)$, we have the following relation:

$$\delta_{\underline{k}_1}(diag(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \delta_{\underline{k}_2}(diag(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) = \delta_{\underline{k}}(diag(p^{\alpha_1}, \dots, p^{\alpha_n})) \quad (10)$$

Note : 2 We have the following relation for the modulus character,

$$\begin{aligned}\delta_B^{-1/2}(diag(p^{\alpha_1}, \dots, p^{\alpha_n})) &= |p^{\alpha_1}|^{\frac{1-n}{2}} \cdots |p^{\alpha_{n_1}}|^{\frac{2n_1-1-n}{2}} |p^{\alpha_{n_1+1}}|^{\frac{2n_1+1-n}{2}} \cdots |p^{\alpha_n}|^{\frac{n-1}{2}} \\ &= \delta_{B_1}^{-1/2}(diag(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \delta_{B_2}^{-1/2}(diag(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) \\ &\quad (p^{\alpha_1} \cdots p^{\alpha_{n_1}})^{\frac{n_2}{2}} (p^{\alpha_{n_1+1}} \cdots p^{\alpha_n})^{-\frac{n_1}{2}}.\end{aligned}$$

Hence we have,

$$\begin{aligned} \delta_{B_1}^{-1/2}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \delta_{B_2}^{-1/2}(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) p^{\frac{n_2}{2}(\alpha_1 + \dots + \alpha_{n_1}) - \frac{n_1}{2}(\alpha_{n_1+1} + \dots + \alpha_n)} \\ = \delta_B^{-1/2}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_n})) \end{aligned} \quad (11)$$

For $\alpha_i \in \mathbb{Z}$, we have,

$$\begin{aligned} \chi_p(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_n})) &= \psi_p(T_1^{\alpha_1} \dots T_n^{\alpha_n}) \\ &= \chi_{p,1} \otimes \chi_{p,2}(T_1^{\alpha_1} \dots T_n^{\alpha_n}) \\ &= \chi_{p,1} \otimes \chi_{p,2}((\mu_1(\text{Frob}_p) p^{\frac{n_2}{2}} X_1)^{\alpha_1} \dots (\mu_2(\text{Frob}_p) p^{-\frac{n_1}{2}} Y_{n_2})^{\alpha_n}) \\ &= \chi_{p,1} \otimes \chi_{p,2}(\mu_1(\text{Frob}_p)^{\alpha_1 + \dots + \alpha_{n_1}} \mu_2(\text{Frob}_p)^{\alpha_{n_1+1} + \dots + \alpha_n} \\ &\quad p^{\frac{n_2}{2}(\alpha_1 + \dots + \alpha_{n_1}) - \frac{n_1}{2}(\alpha_{n_1+1} + \dots + \alpha_n)} X_1^{\alpha_1} \dots X_{n_1}^{\alpha_{n_1}} Y_1^{\alpha_{n_1+1}} \dots Y_{n_2}^{\alpha_n}) \\ &= \mu_1(\text{Frob}_p)^{\alpha_1 + \dots + \alpha_{n_1}} \mu_2(\text{Frob}_p)^{\alpha_{n_1+1} + \dots + \alpha_n} p^{\frac{n_2}{2}(\alpha_1 + \dots + \alpha_{n_1}) - \frac{n_1}{2}(\alpha_{n_1+1} + \dots + \alpha_n)} \\ &\quad \chi_{p,1}(X_1^{\alpha_1} \dots X_{n_1}^{\alpha_{n_1}}) \chi_{p,2}(Y_1^{\alpha_{n_1+1}} \dots Y_{n_2}^{\alpha_n}) \\ &= \mu_1(\text{Frob}_p)^{\alpha_1 + \dots + \alpha_{n_1}} \mu_2(\text{Frob}_p)^{\alpha_{n_1+1} + \dots + \alpha_n} p^{\frac{n_2}{2}(\alpha_1 + \dots + \alpha_{n_1}) - \frac{n_1}{2}(\alpha_{n_1+1} + \dots + \alpha_n)} \\ &\quad \chi_{p,1}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \chi_{p,2}(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) \\ &= \mu_1(\text{Frob}_p)^{\alpha_1 + \dots + \alpha_{n_1}} \mu_2(\text{Frob}_p)^{\alpha_{n_1+1} + \dots + \alpha_n} p^{\frac{n_2}{2}(\alpha_1 + \dots + \alpha_{n_1}) - \frac{n_1}{2}(\alpha_{n_1+1} + \dots + \alpha_n)} \\ &\quad \gamma_1(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \gamma_2(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) \\ &\quad \delta_{B_1}^{-1/2}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \delta_{B_2}^{-1/2}(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) \\ &\quad \delta_{\underline{k}_1}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \delta_{\underline{k}_2}(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) \\ &= \mu_1(\text{Frob}_p)^{\alpha_1 + \dots + \alpha_{n_1}} \mu_2(\text{Frob}_p)^{\alpha_{n_1+1} + \dots + \alpha_n} \gamma_1(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \\ &\quad \gamma_2(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})) \delta_B^{-1/2}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_n})) \\ &\quad \delta_{\underline{k}}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_n})) \\ &= \gamma \delta_B^{-1/2} \delta_{\underline{k}}(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_n})), \end{aligned}$$

where $\gamma : U \rightarrow \mathbb{C}^*$ is the character given by,

$$\begin{aligned} \gamma(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_n})) &= \mu_1(\text{Frob}_p)^{\alpha_1 + \dots + \alpha_{n_1}} \mu_2(\text{Frob}_p)^{\alpha_{n_1+1} + \dots + \alpha_n} \gamma_1(\text{diag}(p^{\alpha_1}, \dots, p^{\alpha_{n_1}})) \\ &\quad \gamma_2(\text{diag}(p^{\alpha_{n_1+1}}, \dots, p^{\alpha_n})). \end{aligned}$$

Since $\chi_{p|U}$ is of the form $\gamma \delta_B^{-1/2} \delta_{\underline{k}}$, it appears as a system of eigenvalue for \mathcal{A}_p in M_x and the refinement it corresponds to is given by γ .

If $\mathcal{R}_1 = (\phi_1, \dots, \phi_{n_1})$ and $\mathcal{R}_2 = (\phi'_1, \dots, \phi'_{n_2})$, then $\phi_i = \gamma_1(\text{diag}(1, \dots, 1, p, 1, \dots, 1))$ and $\phi'_j = \gamma_2(\text{diag}(1, \dots, 1, p, 1, \dots, 1))$. Thus we have,

$$\gamma(\text{diag}(1, \dots, 1, p, 1, \dots, 1)) = \begin{cases} \gamma_1(\text{diag}(1, \dots, 1, p, 1, \dots, 1)) \mu_1(\text{Frob}_p) & \text{if } 1 \leq i \leq n_1, \\ \gamma_2(\text{diag}(1, \dots, 1, p, 1, \dots, 1)) \mu_2(\text{Frob}_p) & \text{if } n_1 < i \leq n. \end{cases}$$

Hence we see that,

$$\chi(\text{diag}(1, \dots, 1, p, 1, \dots, 1)) = \begin{cases} \phi_i \mu_1(\text{Frob}_p) & \text{if } 1 \leq i \leq n_1, \\ \phi'_{i-n_1} \mu_2(\text{Frob}_p) & \text{if } n_1 < i \leq n. \end{cases}$$

So γ corresponds to the refinement given by,

$$\begin{aligned}\mathcal{R} &= (\phi_1\mu_1(Frob_p), \dots, \phi_{n_1}\mu_1(Frob_p), \phi'_1\mu_2(Frob_p), \dots, \phi'_{n_2}\mu_2(Frob_p)) \\ &= (\mu_1((Frob_p)\mathcal{R}_1, \mu_2(Frob_p)\mathcal{R}_2)).\end{aligned}$$

We see that, χ_p corresponds to (π_p, \mathcal{R}) , where π_p is the endoscopic transfer of $\pi_{p,1}$ and $\pi_{p,2}$, since as in the proof of the lemma 4.6, π_p the endoscopic transfer of $\pi_{p,1}$ and $\pi_{p,2}$ has Langlands parameter given by $(\phi_1\mu_1(Frob_p), \dots, \phi_{n_1}\mu_1(Frob_p), \phi'_1\mu_2(Frob_p), \dots, \phi'_{n_2}\mu_2(Frob_p))$. By definition of χ' , χ'_p corresponds to (π_p, \mathcal{R}) . Thus $\chi'_p = \chi_p$. □

Thus we obtain a map, $f = \beta \circ \alpha : \mathcal{E}_{n_1} \times \mathcal{E}_{n_2} \rightarrow \mathcal{E}_n$. Let $((\pi_1, \mathcal{R}_1), (\pi_2, \mathcal{R}_2))$ be a point in $\mathcal{E}_{n_1} \times \mathcal{E}_{n_2}$, which corresponds to $\mathcal{H}_1 \otimes \mathcal{H}_2$ system of eigenvalue $\chi_1 \otimes \chi_2$ in $(M'_k)^{cl}$. Under the map α , $\chi_1 \otimes \chi_2$ maps to \mathcal{H} system of eigenvalue $\chi = (\chi_1 \otimes \chi_2)|_{\mathcal{H}}$ in $(M'_k)^{cl}$. Under the map β , χ maps to \mathcal{H} system of eigenvalue $\chi' = \chi$ appearing in $(M'_k)^{cl}$, which corresponds to (π, \mathcal{R}) , where π is the endoscopic transfer of π_1 and π_2 and $\mathcal{R} = (\mu_1((Frob_p)\mathcal{R}_1, \mu_2(Frob_p)\mathcal{R}_2))$. □

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